

On the splitting problem for complex homogeneous supermanifolds ¹

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ABSTRACT. It is well-known that any Lie supergroup \mathcal{G} is split, i.e. its structure sheaf is isomorphic to the structure sheaf of a certain vector bundle. However, there are non-split complex homogeneous supermanifolds. We describe left invariant gradings of a complex homogeneous supermanifold \mathcal{G}/\mathcal{H} induced by gradings of the Lie supergroup \mathcal{G} in terms of so called split grading operators. Sufficient conditions for a homogeneous supermanifold to be split are given in terms of Lie superalgebras and Lie subsuperalgebras.

1. Introduction

A supermanifold is called split if its structure sheaf is isomorphic to the exterior power of a certain vector bundle. It is well-known that any real supermanifold is split. However, it is false in the complex analytic case. The property of a supermanifold to be split is very important for several problems. For instance, if a supermanifold \mathcal{M} is projective, then, obviously, its underlying space is also projective. In [10] it was mentioned that almost all flag supermanifolds are not projective, hence, the converse statement is in general false. However, as was shown in [5] any split supermanifold with the projective underlying space is projective. Another problem, when the property of a supermanifold to be split is very important, is the calculation of the cohomology group with values in a vector bundle on a supermanifold. In the split case we may use the well elaborated tools of complex analytic geometry. In the general case, several methods were suggested by Onishchik's school: spectral sequences, see e.g. [9]. All these methods connect the cohomology group with values in a vector bundle with the cohomology group with values in the corresponding split vector bundle.

How to find out, whether a complex supermanifold is split or non-split? Several methods were suggested by Green, Onishchik and Koszul. In [2] Green described a moduli space with a marked point such that any non-marked point corresponds to a non-split supermanifold while the marked point corresponds to a split supermanifold. The calculation of the Green moduli space is a difficult problem itself, so the method is difficult to apply. Furthermore, for calculation of holomorphic vector fields over super-Grassmannians Onishchik and Serov [6, 7, 8] considered so called grading

¹Supported by Max Planck Institute for Mathematics Bonn.

derivations, which corresponds to \mathbb{Z} -gradings of the structure sheaf of a supermanifold. For instance, it was shown that almost all super-Grassmannians do not possess any grading derivations, (i.e. their structure sheaves do not possess \mathbb{Z} -gradings), hence, they are non-split. In [3] the following statement was proved: if the tangent bundle of a supermanifold \mathcal{M} possesses a (holomorphic) connection, then \mathcal{M} is split. In fact, it was shown that we can assign the grading derivation to any supermanifold with a connection, and that this grading derivation is induced by \mathbb{Z} -grading of a vector bundle.

In the present paper a \mathbb{Z} -grading of the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a supermanifold \mathcal{M} , which corresponds to a vector bundle, is called a split grading. A split grading is called left invariant if it is invariant with respect to all left translations. The purpose of our paper is to describe grading operators corresponding to left invariant split gradings on a homogeneous superspace \mathcal{G}/\mathcal{H} which are compatible with gradings on \mathcal{G} .

2. Lie supergroups and homogeneous spaces

We will use the word "supermanifold" in the sense of Berezin and Leites, see [1]. We will denote a supermanifold by \mathcal{M} , its base by \mathcal{M}_0 and its structure sheaf by $\mathcal{O}_{\mathcal{M}}$. If $p : \mathcal{M} \rightarrow \mathcal{N}$ is a morphism of supermanifolds, then we denote by p_0 the corresponding morphism of the bases $\mathcal{M}_0 \rightarrow \mathcal{N}_0$ and by p^* the morphisms of sheaves $\mathcal{O}_{\mathcal{N}} \rightarrow (p_0)_*(\mathcal{O}_{\mathcal{M}})$.

2.1 Lie supergroups and super Harish-Chandra pairs. A Lie supergroup is a group object in the category of supermanifolds, i.e., a supermanifold \mathcal{G} with three morphisms: the multiplication morphism, the inversion morphism and the identity morphism, which satisfy the usual conditions, modeling the group axioms. It is well known that the structure sheaf of a Lie supergroup and the supergroup morphisms can be explicitly described in terms of the corresponding Lie superalgebra using so-called super Harish-Chandra pairs, see [11]. A *super Harish-Chandra pair* is a pair (G, \mathfrak{g}) that consists of a Lie group G and a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_0 = \text{Lie } G$, provided with a representation $\text{Ad} : G \rightarrow \text{Aut } \mathfrak{g}$ of G in \mathfrak{g} such that: 1. Ad preserves the parity and induces the adjoint representation of G in \mathfrak{g}_0 ; 2. the differential $(d \text{Ad})_e$ at the identity $e \in G$ coincides with the adjoint representation ad of \mathfrak{g}_0 in \mathfrak{g} .

If a super Harish-Chandra pair (G, \mathfrak{g}) is given, it determines the Lie supergroup \mathcal{G} in the following way, see [4]. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping superalgebra of \mathfrak{g} . It is clear that $\mathcal{U}(\mathfrak{g})$ is a $\mathcal{U}(\mathfrak{g}_0)$ -module, where $\mathcal{U}(\mathfrak{g}_0)$ is the universal enveloping algebra of \mathfrak{g}_0 . Denote by $\mathcal{F}_{\mathcal{G}_0}$ the structure sheaf of

the manifold \mathcal{G}_0 . The natural action of $\mathfrak{g}_{\bar{0}}$ on the sheaf $\mathcal{F}_{\mathcal{G}_0}$ gives rise to a structure of $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$ -module on $\mathcal{F}_{\mathcal{G}_0}(U)$ for any open set $U \subset \mathcal{G}_0$. Putting

$$\mathcal{O}_{\mathcal{G}}(U) = \text{Hom}_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}(\mathfrak{U}(\mathfrak{g}), \mathcal{F}_{\mathcal{G}_0}(U))$$

for every open $U \subset \mathcal{G}_0$, we get a sheaf $\mathcal{O}_{\mathcal{G}}$ of \mathbb{Z}_2 -graded vector spaces (here we assume that the functions from $\mathcal{F}_{\mathcal{G}_0}(U)$ are even). The enveloping superalgebra $\mathfrak{U}(\mathfrak{g})$ has a Hopf superalgebra structure. Using this structure we can define the product of elements from $\mathcal{O}_{\mathcal{G}}$ such that $\mathcal{O}_{\mathcal{G}}$ becomes a sheaf of superalgebras, see [4] for details. A supermanifold structure on $\mathcal{O}_{\mathcal{G}}$ is determined by the isomorphism $\Phi_{\mathfrak{g}} : \mathcal{O}_{\mathcal{G}} \rightarrow \text{Hom}(\bigwedge(\mathfrak{g}_{\bar{1}}), \mathcal{F}_{\mathcal{G}_0})$, $f \mapsto f \circ \gamma_{\mathfrak{g}}$, where

$$\gamma_{\mathfrak{g}} : \bigwedge(\mathfrak{g}_{\bar{1}}) \rightarrow \mathfrak{U}(\mathfrak{g}), \quad X_1 \wedge \cdots \wedge X_r \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^{|\sigma|} X_{\sigma(1)} \cdots X_{\sigma(r)}. \quad (1)$$

The following formulas define the multiplication morphism, the inversion morphism and the identity morphism respectively:

$$\begin{aligned} \mu^*(f)(X \otimes Y)(g, h) &= f(\text{Ad}(h^{-1})(X) \cdot Y)(gh); \\ \iota^*(f)(X)(g) &= f(\text{Ad}(g)(S(X)))(g^{-1}); \\ \varepsilon^*(f) &= f(1)(e). \end{aligned} \quad (2)$$

Here $X, Y \in \mathfrak{U}(\mathfrak{g})$, $f \in \mathcal{O}_{\mathcal{G}}$, $g, h \in \mathcal{G}_0$ and S is the antipode map of the Hopf superalgebra $\mathfrak{U}(\mathfrak{g})$. Here we identify the enveloping superalgebra $\mathfrak{U}(\mathfrak{g} \oplus \mathfrak{g})$ with the tensor product $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$.

We will identify the Lie superalgebra \mathfrak{g} of a Lie supergroup \mathcal{G} with the tangent space $T_e(\mathcal{G})$ at $e \in \mathcal{G}_0$. If $T \in \mathfrak{g}$, the corresponding left invariant vector field is given by

$$(\text{id} \otimes T) \circ \mu^*, \quad (3)$$

where μ is the multiplication morphism of \mathcal{G} . Recall that this correspondence determines an isomorphism of Lie superalgebras. Denote by l_g (r_g) the left (or right) translation with respect to $g \in \mathcal{G}_0$. The morphisms l_g and r_g are given by the following formulas:

$$l_g^*(f)(X)(h) = f(X)(gh); \quad r_g^*(f)(X)(h) = f(\text{Ad}(g^{-1})X)(hg), \quad (4)$$

where $f \in \mathcal{O}_{\mathcal{G}}$, $X \in \mathfrak{U}(\mathfrak{g})$ and $g, h \in \mathcal{G}_0$.

2.2 Homogeneous supermanifolds. A supermanifold is called *homogeneous* if it possesses a transitive action of a certain Lie supergroup. (More precisely, see e.g. [12].) If a supermanifold \mathcal{M} is \mathcal{G} -homogeneous and $\mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M}$ is a

transitive action, then \mathcal{M} is isomorphic to the supermanifold \mathcal{G}/\mathcal{H} , where \mathcal{H} is the isotropy subsupergroup of a certain point. Recall that the underlying space of \mathcal{G}/\mathcal{H} is $\mathcal{G}_0/\mathcal{H}_0$ and the structure sheaf $\mathcal{O}_{\mathcal{G}/\mathcal{H}}$ of \mathcal{G}/\mathcal{H} is given by

$$\mathcal{O}_{\mathcal{G}/\mathcal{H}} = \{f \in \pi_*(\mathcal{O}_{\mathcal{G}}) \mid \mu_{\mathcal{G} \times \mathcal{H}}^*(f) = \text{pr}^*(f)\}, \quad (5)$$

where $\pi : \mathcal{G}_0 \rightarrow \mathcal{G}_0/\mathcal{H}_0$ is the natural map, $\mu_{\mathcal{G} \times \mathcal{H}}$ is the morphism $\mathcal{G} \times \mathcal{H} \rightarrow \mathcal{G}$ induced by the multiplication map in \mathcal{G} and $\text{pr} : \mathcal{G} \times \mathcal{H} \rightarrow \mathcal{G}$ is the projection. Using (2) we can rewrite (5) in the following way:

$$\mathcal{O}_{\mathcal{G}/\mathcal{H}} = \left\{ f \in \pi_*(\mathcal{O}_{\mathcal{G}}) \mid f(\text{Ad}(h^{-1})(X)Y)(gh) = \begin{cases} f(X)(g), & Y \in \mathbb{C}; \\ 0, & Y \notin \mathbb{C}. \end{cases} \right\},$$

where $X \in \mathfrak{U}(\mathfrak{g})$, $Y \in \mathfrak{U}(\mathfrak{h})$, $\mathfrak{h} = \text{Lie } \mathcal{H}$, $g \in \mathcal{G}_0$ and $h \in \mathcal{H}_0$. Now it is easy to see that $f \in \mathcal{O}_{\mathcal{G}/\mathcal{H}}$ if f is \mathcal{H}_0 -right invariant, i.e. $r_h^*(f) = f$ for any $h \in \mathcal{H}_0$, and $Y(f) = 0$ for all $Y \in \mathfrak{h}_{\bar{1}}$, where $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$.

Sometimes we will consider also the left action $\mathcal{H} \times \mathcal{G} \rightarrow \mathcal{G}$ of a subsupergroup \mathcal{H} on a Lie supergroup \mathcal{G} . The corresponding quotient supermanifold we will denote by $\mathcal{G} \backslash \mathcal{H}$.

2.3 Split supermanifolds. A supermanifold \mathcal{M} is called *split* if $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$, where \mathcal{E} is a sheaf of sections of a certain vector bundle. In this case $\mathcal{O}_{\mathcal{M}}$ possesses the \mathbb{Z} -grading induced by the natural \mathbb{Z} -grading of $\bigwedge \mathcal{E} = \bigoplus_p \bigwedge^p \mathcal{E}$. Such gradings of $\mathcal{O}_{\mathcal{M}}$ we will call *split gradings*.

Example 1. It is well-known that any Lie supergroup \mathcal{G} is split. Indeed, the underlying space \mathcal{G}_0 is a Lie subsupergroup of \mathcal{G} , hence, there exists the homogeneous space $\mathcal{G}/\mathcal{G}_0$, which is isomorphic to the supermanifold \mathcal{N} such that \mathcal{N}_0 is a point $\text{pt} = \mathcal{G}_0/\mathcal{G}_0$ and $\mathcal{O}_{\mathcal{N}} \simeq \bigwedge(m)$, where $m = \dim \mathfrak{g}_{\bar{1}}$. By definition, the structure sheaf $\mathcal{O}_{\mathcal{N}}$ consists of all r_g -invariant functions, $g \in \mathcal{G}_0$. We have the natural map $p = (p_0, p^*) : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{G}_0$, where $p_0 : \mathcal{G}_0 \rightarrow \text{pt}$ and $p^* : \mathcal{O}_{\mathcal{N}} \rightarrow \mathcal{O}_{\mathcal{G}}$ is the natural inclusion. It is well known that $p : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{G}_0$ is a principal bundle. Using the fact that the underlying space of $\mathcal{G}/\mathcal{G}_0$ is a point we get $\mathcal{G} \simeq \mathcal{N} \times \mathcal{G}_0$.

This statement follows also from the fact that any Lie supergroup is determined by its super Harish-Chandra pair. A different proof of this statement is given also in [3].

Example 2. As an example of a homogeneous non-split supermanifold we can cite the super-grassmannian $\mathbf{Gr}_{m|n,r|s}$ for $0 < r < m$ and $0 < s < n$.

Denote by **SSM** the category of split supermanifolds. Objects **Ob SSM** of this category are all split supermanifolds \mathcal{M} with a fixed split grading. Further, if $X, Y \in \mathbf{Ob} \text{ SSM}$, we put

$$\text{Hom}(X, Y) = \begin{array}{l} \text{all morphisms of } X \text{ to } Y \\ \text{preserving the split gradings.} \end{array}$$

As in the category of supermanifolds, we can define in **SSM** a group object (*split Lie supergroup*), an action of a split Lie supergroup on a split supermanifold (*split action*) and a *split homogeneous supermanifold*.

There is a functor gr from the category of supermanifolds to the category of split supermanifolds. Let us briefly describe this construction. Let \mathcal{M} be a supermanifold. Denote by $\mathcal{J}_{\mathcal{M}} \subset \mathcal{O}_{\mathcal{M}}$ the subsheaf of ideals generated by odd elements of $\mathcal{O}_{\mathcal{M}}$. Then by definition $\text{gr } \mathcal{M}$ is the split supermanifold with the structure sheaf

$$\text{gr } \mathcal{O}_{\text{gr } \mathcal{M}} = \bigoplus_{p \geq 0} (\text{gr } \mathcal{O}_{\text{gr } \mathcal{M}})_p, \quad \mathcal{J}_{\mathcal{M}}^0 := \mathcal{O}_{\mathcal{M}}, \quad (\text{gr } \mathcal{O}_{\text{gr } \mathcal{M}})_p := \mathcal{J}_{\mathcal{M}}^p / \mathcal{J}_{\mathcal{M}}^{p+1}.$$

In this case $(\text{gr } \mathcal{O}_{\text{gr } \mathcal{M}})_1$ is a locally free sheaf and there is a natural isomorphism of $\text{gr } \mathcal{O}_{\mathcal{M}}$ onto $\bigwedge (\text{gr } \mathcal{O}_{\mathcal{M}})_1$. If $\psi = (\psi_0, \psi^*) : \mathcal{M} \rightarrow \mathcal{N}$ is a morphism, then $\text{gr}(\psi) = (\psi_0, \text{gr}(\psi^*)) : \text{gr } \mathcal{M} \rightarrow \text{gr } \mathcal{N}$ is defined by

$$\text{gr}(\psi^*)(f + \mathcal{J}_{\mathcal{N}}^p) := \psi^*(f) + \mathcal{J}_{\mathcal{M}}^p \text{ for } f \in (\mathcal{J}_{\mathcal{N}})^{p-1}.$$

Recall that by definition every morphism of supermanifolds is even and as a consequence sends $\mathcal{J}_{\mathcal{N}}^p$ into $\mathcal{J}_{\mathcal{M}}^p$.

2.4 Split Lie supergroups. Let \mathcal{G} be a Lie supergroup with the group morphisms μ , ι and ε : the multiplication, the inversion and the identity, respectively. Then $\text{gr } \mathcal{G}$ is a split Lie supergroup with the group morphisms $\text{gr}(\mu)$, $\text{gr}(\iota)$ and $\text{gr}(\varepsilon)$. Let us describe the supergroup $\text{gr } \mathcal{G}$ in terms of super Harish-Chandra pairs. The universal enveloping superalgebra $\mathfrak{U}(\mathfrak{g})$ possesses the following filtration:

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g})_{(m)} \supset \mathfrak{U}(\mathfrak{g})_{(m-1)} \supset \cdots \supset \mathfrak{U}(\mathfrak{g})_{(0)} \supset \{0\}, \quad (6)$$

where $m = \dim \mathfrak{g}_{\bar{1}}$ and $\mathfrak{U}(\mathfrak{g})_{(p)}$ is the vector subspace generated by $\{Y \cdot X_1 \cdot \cdots \cdot X_q, q \leq k\}$, where $Y \in \mathfrak{U}(\mathfrak{g}_0)$, $X_i \in \mathfrak{g}_{\bar{1}}$. This filtration induces the filtration in $\mathcal{O}_{\mathcal{G}} = \text{Hom}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), \mathcal{F}_{\mathcal{G}_0})$:

$$\mathcal{O}_{\mathcal{G}} = (\mathcal{O}_{\mathcal{G}})_{(-1)} \supset (\mathcal{O}_{\mathcal{G}})_{(0)} \supset \cdots,$$

where

$$(\mathcal{O}_{\mathcal{G}})_{(p)} := \{f \in \mathcal{O}_{\mathcal{G}} \mid f|_{\mathfrak{U}(\mathfrak{g})_{(p)}} = 0\}.$$

Clearly, $(\mathcal{O}_{\mathcal{G}})_{(p)} = \mathcal{J}^{p+1}$, hence,

$$\text{gr } \mathcal{O}_{\mathcal{G}} \simeq \text{Hom}_{\mathfrak{U}(\mathfrak{g}_0)}(\text{gr } \mathfrak{U}(\mathfrak{g}), \mathcal{F}_{\mathcal{G}_0}) \simeq \mathcal{O}_{\mathcal{G}'} := \text{Hom}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}'), \mathcal{F}_{\mathcal{G}_0}),$$

where $\text{gr } \mathfrak{U}(\mathfrak{g})$ is the graded superalgebra corresponding to the filtration (6), \mathcal{G}' is the Lie supergroup corresponding to the super Harish-Chandra pair

$(\mathcal{G}_0, \mathfrak{g}')$ and \mathfrak{g}' is the following Lie superalgebra: $\mathfrak{g}' \simeq \mathfrak{g}$ as vector superspaces and the Lie bracket is defined by

$$[X, Y] = \begin{cases} [X, Y], & \text{if } X, Y \in \mathfrak{g}_{\bar{0}} \text{ or } X \in \mathfrak{g}_{\bar{0}} \text{ and } Y \in \mathfrak{g}_{\bar{1}}; \\ 0, & \text{if } X, Y \in \mathfrak{g}_{\bar{1}}. \end{cases} \quad (7)$$

The Lie supergroup \mathcal{G}' is a split Lie supergroup with respect to the grading induced by the isomorphism $\Phi_{\mathfrak{g}'}$. It is easy to see that $\text{gr } \mathcal{G} \simeq \mathcal{G}'$ as Lie supergroups. Denote by μ' , ι' and ε' the multiplication, inversion and identity morphisms of \mathcal{G}' , respectively. Let us describe these morphisms more precisely.

A direct calculation leads to the following lemma:

Lemma 1. *Let \mathfrak{g} be a Lie superalgebra and $X_i, Y_j \in \mathfrak{g}_{\bar{1}}$, $i = 1, \dots, r$, $j = 1, \dots, s$. Assume that $[X_i, Y_j] = 0$ for any i, j . Then we have*

$$\gamma_{\mathfrak{g}}(X_1 \wedge \dots \wedge X_r \wedge Y_1 \wedge \dots \wedge Y_s) = \gamma_{\mathfrak{g}}(X_1 \wedge \dots \wedge X_r) \cdot \gamma_{\mathfrak{g}}(Y_1 \wedge \dots \wedge Y_s),$$

where $\gamma_{\mathfrak{g}}$ is given by (1). \square

From Lemma 1 it follows:

Proposition 1. *If we identify the supermanifold $\text{gr } \mathcal{G} \simeq \mathcal{G}'$ with the supermanifold $\text{Hom}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0})$ using the isomorphism $f \mapsto f \circ \gamma_{\mathfrak{g}'}$, then the supergroup morphisms of $\text{gr } \mathcal{G}$ are given by*

$$\begin{aligned} \mu'^*(f)(X \wedge Y)(g, h) &= f(\text{Ad}(h^{-1})(X) \wedge Y)(gh); \\ \iota'^*(f)(X)(g) &= f(\text{Ad}(g)(S(X)))(g^{-1}); \\ \varepsilon'^*(f) &= f(1)(e). \end{aligned} \quad (8)$$

Here $X, Y \in \bigwedge(\mathfrak{g}_{\bar{1}})$, $f \in \text{Hom}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0})$, $g, h \in \mathcal{G}_0$ and S is the antipode map of $\bigwedge(\mathfrak{g}_{\bar{1}})$. \square

3. Split grading operators

Let \mathcal{M} be a supermanifold, $\text{gr } \mathcal{M}$ the corresponding split supermanifold, \mathcal{I} be the sheaf of ideals generated by odd elements of $\mathcal{O}_{\mathcal{M}}$, $\mathcal{T} = \text{Der } \mathcal{O}_{\mathcal{M}}$ the tangent sheaf of \mathcal{M} and $\text{gr } \mathcal{T} = \text{Der}(\mathcal{O}_{\text{gr } \mathcal{M}})$ the tangent sheaf of $\text{gr } \mathcal{M}$. The sheaf \mathcal{T} is naturally \mathbb{Z}_2 -graded and the sheaf $\text{gr } \mathcal{T}$ is naturally \mathbb{Z} -graded; the gradings are induced by the \mathbb{Z}_2 - and \mathbb{Z} -grading of $\mathcal{O}_{\mathcal{M}}$ and $\text{gr } \mathcal{O}_{\mathcal{M}}$ respectively. We have the exact sequence

$$0 \rightarrow \mathcal{T}_{(2)\bar{0}} \rightarrow \mathcal{T}_{\bar{0}} \xrightarrow{\alpha} (\text{gr } \mathcal{T})_0 \rightarrow 0, \quad (9)$$

where

$$\mathcal{T}_{(2)\bar{0}} = \{v \in \mathcal{T}_{\bar{0}} \mid v(\mathcal{O}_{\mathcal{M}}) \subset \mathcal{J}^2\}$$

and α is the composition of the natural map $\mathcal{T}_{\bar{0}} \rightarrow \mathcal{T}_{\bar{0}}/\mathcal{T}_{(2)\bar{0}}$ and the following isomorphism

$$\mathcal{T}_{\bar{0}}/\mathcal{T}_{(2)\bar{0}} \rightarrow (\text{gr } \mathcal{T})_0, [w] \mapsto \tilde{w}, \tilde{w}(f + \mathcal{J}^{p+1}) := w(f) + \mathcal{J}^{p+1},$$

where $w \in \mathcal{T}_{\bar{0}}$, $[w]$ is the image of w in $\mathcal{T}_{\bar{0}}/\mathcal{T}_{(2)\bar{0}}$ and $f \in \mathcal{J}^p$.

Assume that the sheaf $\mathcal{O}_{\mathcal{M}}$ is \mathbb{Z} -graded, i.e. $\mathcal{O}_{\mathcal{M}} = \bigoplus_p (\mathcal{O}_{\mathcal{M}})_p$. Then we have the operator w defined by $w(f) = pf$, where $f \in \mathcal{O}_p$. Such operators are called *grading operators*. We will call a grading operator w on \mathcal{M} a *split grading operator* if it corresponds to a split grading of $\mathcal{O}_{\mathcal{M}}$. The sheaf $\text{gr } \mathcal{O}_{\mathcal{M}}$ is naturally \mathbb{Z} -graded. Denote by a the corresponding graded operator; it is an even vector field on $\text{gr } \mathcal{M}$.

Lemma 2. *A supermanifold \mathcal{M} is split iff $a \in \text{Im } H^0(\alpha)$, where*

$$H^0(\alpha) : H^0(\mathcal{M}_0, \mathcal{T}_{\bar{0}}) \rightarrow H^0(\mathcal{M}_0, (\text{gr } \mathcal{T})_0).$$

Proof. The statement of the proposition can be deduced from the following observation doing by Koszul in [3], Section 1. Let A be a superalgebra over \mathbb{C} (in [3] the author considered more general algebras). Let \mathfrak{m} be a nilpotent ideal in A . An even derivation w is called *adapted* to the filtration $A \supset \mathfrak{m} \supset \mathfrak{m}^2 \dots$ if

$$(w - r \text{id})\mathfrak{m}^r \subset \mathfrak{m}^{r+1} \text{ for any } r \geq 0.$$

Denote by $D_{\mathfrak{m}}^{ad}$ the set of all derivations adapted to \mathfrak{m} . In [3] it was shown that $D_{\mathfrak{m}}^{ad}$ is not empty iff the filtration of A is splittable.

In our case we consider the sheaf of superalgebras $\mathcal{O}_{\mathcal{M}}$ and its subsheaf of ideals \mathcal{J} except for A and \mathfrak{m} , respectively. The set $D_{\mathfrak{m}}^{ad}$ is in our case the set of global derivations of $\mathcal{O}_{\mathcal{M}}$ adapted to the filtration $\mathcal{O}_{\mathcal{M}} \supset \mathcal{J} \supset \dots$. Clearly, $D_{\mathfrak{m}}^{ad}$ is not empty iff $a \in \text{Im } H^0(\alpha)$. \square

Example 3. Consider the supermanifold $\mathcal{G}/\mathcal{G}_0$; its structure sheaf is isomorphic to $\bigwedge(\mathfrak{g}_{\bar{1}})$ (compare with Example 1). Denote by (ε_i) the system of odd (global) coordinates on $\mathcal{G}/\mathcal{G}_0$. An example of a split grading operator on the Lie supergroup \mathcal{G} is $\sum \varepsilon^i X_i$. Here (X_i) is a basis of odd left invariant vector fields on \mathcal{G} such that $X_i(\varepsilon^j)(e) = \delta_i^j$. We may produce other examples if we use right invariant vector fields or odd (global) coordinates on $\mathcal{G}/\mathcal{G}_0$.

Denote by $\text{Aut}_{(2)} \mathcal{O}_{\mathcal{M}}$ the group of automorphisms a of $\mathcal{O}_{\mathcal{M}}$ such that $\text{gr } a = \text{id}$.

Proposition 2. [Koszul] *If \mathcal{M} is a split supermanifold and w a certain split grading operator on \mathcal{M} , then any other split grading on \mathcal{M} is given by $a \circ w \circ a^{-1}$, where $a \in \text{Aut}_{(2)} \mathcal{O}_{\mathcal{M}}$.*

In particular, any split grading operator on a Lie supergroup \mathcal{G} is given by $\sum \varepsilon^i X_i + \chi$, where $\sum \varepsilon^i X_i$ is as in Example 3 and $\chi \in H^0(\mathcal{G}_0, \mathcal{T}_{(2)\bar{0}})$ is any vector field.

Proof. See Proposition 1.2 in [3]. \square

4. Compatible split gradings on \mathcal{G}/\mathcal{H}

The aim of our paper is the study of split gradings on a homogeneous superspace $\mathcal{M} = \mathcal{G}/\mathcal{H}$ induced by certain split gradings on Lie supergroup \mathcal{G} .

Definition. A split grading on a Lie supergroup $\mathcal{O}_{\mathcal{G}_0} = \sum (\mathcal{O}_{\mathcal{G}})_p$ is called *compatible* with $\mathcal{O}_{\mathcal{M}} \subset \mathcal{O}_{\mathcal{G}}$ if the following holds:

$$f \in \mathcal{O}_{\mathcal{M}} \Rightarrow f_p \in \mathcal{O}_{\mathcal{M}} \text{ for all } p,$$

where $f = \sum f_p$ and $f_p \in (\mathcal{O}_{\mathcal{G}})_p$.

It is not clear from the definition that the grading $(\mathcal{O}_{\mathcal{M}})_p = \mathcal{O}_{\mathcal{M}} \cap (\mathcal{O}_{\mathcal{G}})_p$ of $\mathcal{O}_{\mathcal{M}}$ is a split grading on $\mathcal{O}_{\mathcal{M}}$. However, the following holds.

Proposition 3. *A compatible with $\mathcal{O}_{\mathcal{M}}$ split grading on $\mathcal{O}_{\mathcal{G}}$ is a split grading on $\mathcal{O}_{\mathcal{M}}$.*

Proof. The idea of the proof is to show that $\alpha(\sum \varepsilon^i X_i + \chi) = a$, where $\sum \varepsilon^i X_i + \chi$ is as in Proposition 2 and a is as in Lemma 2 for $\text{gr } \mathcal{M}$.

First of all note that $\text{gr } \mathcal{M} \simeq \text{gr } \mathcal{G}/\text{gr } \mathcal{H}$, see [12], Theorem 3. Using (5) and (8) we see that

$$\mathcal{O}_{\text{gr } \mathcal{M}} \simeq \text{Hom} \left(\bigwedge (\mathfrak{g}_{\bar{1}}) / \bigwedge (\mathfrak{g}_{\bar{1}}) \mathfrak{h}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0} \right)^{\mathcal{H}_0}.$$

as split supermanifolds. Here the split grading on the second sheaf is given by:

$$\text{Hom} \left(\bigwedge (\mathfrak{g}_{\bar{1}}) / \bigwedge (\mathfrak{g}_{\bar{1}}) \mathfrak{h}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0} \right)^{\mathcal{H}_0} \simeq \bigoplus_p \text{Hom} \left(\bigwedge^p (\mathfrak{g}_{\bar{1}}) / \bigwedge^{p-1} (\mathfrak{g}_{\bar{1}}) \mathfrak{h}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0} \right)^{\mathcal{H}_0}.$$

Denote by w the split grading operator on \mathcal{G} corresponding to a compatible with $\mathcal{O}_{\mathcal{M}}$ split grading on $\mathcal{O}_{\mathcal{G}}$. By Proposition 2 we have $w = \sum \varepsilon^i X_i + \chi$. Furthermore,

$$\alpha(\sum \varepsilon^i X_i + \chi) = \alpha(\sum \varepsilon^i X_i) = \sum e^i X'_i,$$

where $e^i = \varepsilon^i + \mathcal{J} \in (\mathcal{O}_{\text{gr } \mathcal{G}})_0 \simeq \mathfrak{g}_1^* \subset \text{Hom}(\bigwedge(\mathfrak{g}_1), \mathcal{F}_{\mathcal{G}_0})$ is a basis of \mathfrak{g}_1^* and X'_i are odd left invariant vector fields on $\text{gr } \mathcal{G}$ such that $X'_i(e^j) = \delta_i^j$. Moreover, using (3) and Proposition 1, we see that $X'_i(\text{Hom}(\mathbb{C}, \mathcal{F}_{\mathcal{G}_0})) = \{0\}$ for all i . Therefore, $\sum e^i X'_i$ is the split grading operator on $\text{gr } \mathcal{G}$ corresponding to the natural split grading

$$\mathcal{O}_{\text{gr } \mathcal{G}} \simeq \bigoplus_p \text{Hom} \left(\bigwedge^p(\mathfrak{g}_1), \mathcal{F}_{\mathcal{G}_0} \right).$$

Furthermore, the operator $\sum e^i X'_i$ leaves $\mathcal{O}_{\text{gr } \mathcal{M}}$ invariant and it is split because $\text{Hom}(\bigwedge(\mathfrak{g}_1)/\bigwedge(\mathfrak{g}_1)\mathfrak{h}_1, \mathcal{F}_{\mathcal{G}_0})^{\mathcal{H}_0}$ is obviously a vector bundle. The result follows from Lemma 2. \square

Denote by v the split grading operator corresponding to a split grading of $\mathcal{O}_{\mathcal{G}}$. It is easy to see that this split grading is compatible with $\mathcal{O}_{\mathcal{M}}$ iff $v(\mathcal{O}_{\mathcal{M}}) \subset \mathcal{O}_{\mathcal{M}}$. Therefore, our aim is to describe all split grading operators on \mathcal{G} which leave the sheaf $\mathcal{O}_{\mathcal{M}}$ invariant. First of all let us consider the situation when a split grading operator w on \mathcal{G} is invariant with respect to a Lie subsupergroup \mathcal{H} , i.e.

$$\begin{cases} r_h^* \circ w = w \circ r_h^*, & \text{for all } h \in \mathcal{H}_0; \\ [Y, w] = 0, & \text{for all } Y \in \mathfrak{h}_1. \end{cases} \quad (10)$$

Here $(\mathcal{H}_0, \mathfrak{h})$ is the super Harish-Chandra pair of \mathcal{H} .

Proposition 4. *If a split grading operator w on \mathcal{G} is invariant with respect to a Lie subsupergroup \mathcal{H} , then $\mathfrak{h} = \mathfrak{h}_0$.*

Proof. By Proposition 2 any split grading operator on \mathcal{G} is given by $w = \sum \varepsilon^i X_i + \chi$. If X is a vector field on \mathcal{G} , denote by $X_e \in T_e(\mathcal{G})$ the corresponding tangent vector at the identity e . Consider the second equation in (10). At the point e , we have

$$[Y, w]_e = \left(\sum Y(\varepsilon^i) X_i - \sum \varepsilon^i Y \circ X_i - \sum \varepsilon^i X_i \circ Y + [Y, \chi] \right)_e = \sum Y(\varepsilon^i)(e)(X_i)_e = 0, \quad Y \in \mathfrak{h}_1.$$

The tangent vectors $(X_i)_e$ form a basis in $T_e(\mathcal{G})$, hence $Y(\varepsilon^i)(e) = 0$ for all i . Therefore, $Y = 0$. The proof is complete. \square

Example 4. It is well known that the supermanifold \mathcal{G}/\mathcal{H} , where \mathcal{H} is an ordinary Lie group, is split. (See [4].)

Consider now more general situation, when a split grading operator w leaves $\mathcal{O}_{\mathcal{M}}$ invariant. Let $f \in \mathcal{O}_{\mathcal{M}}$, then $w(f) \in \mathcal{O}_{\mathcal{M}}$ iff $r_h^*(w(f)) = w(f)$ and

$Y(w(f)) = 0$, where $h \in \mathcal{H}_0$ and $Y \in \mathfrak{h}_{\bar{1}}$. These conditions are equivalent to the following ones:

$$(r_h^* \circ w \circ (r_h^{-1})^* - w)|_{\mathcal{O}_{\mathcal{M}}} = 0; \quad [Y, w]|_{\mathcal{O}_{\mathcal{M}}} = 0. \quad (11)$$

It seems to us that the system (11) is hard to solve in general. Consider now a special type of split grading operators, so called left invariant grading operators.

Definition. A split grading on \mathcal{G} is called *left invariant* if it is invariant with respect to the left translations. More precisely, if $f \in (\mathcal{O}_{\mathcal{G}})_p$ then $l_g^*(f) \in (\mathcal{O}_{\mathcal{G}})_p$ for all $g \in \mathcal{G}_0$.

It is easy to see that a split grading is left invariant iff the corresponding split grading operator w is invariant with respect to left translations, i.e. $l_g^* \circ w = w \circ l_g^*$, $g \in \mathcal{G}_0$. For example, the split grading operator $\sum \varepsilon^i X_i$ constructed in Example 3 is obviously a left invariant grading operator. Let us describe all such operators. We need the following lemma:

Lemma 3. *The map $\Phi_{\mathfrak{g}} : \mathcal{O}_{\mathcal{G}} \rightarrow \text{gr } \mathcal{O}_{\mathcal{G}}$ is invariant with respect to the left and right translations.*

Proof. Denote by r'_h and l'_h , $h \in \mathcal{G}_0$, the right and the left translation in $\text{gr } \mathcal{G}$ respectively. Let us show that

$$(r'_h)^* \circ \Phi_{\mathfrak{g}} = \Phi_{\mathfrak{g}} \circ r_h^*. \quad (12)$$

Indeed, for $Z \in \bigwedge \mathfrak{g}_{\bar{1}}$, $g, h \in \mathcal{G}_0$ we have

$$\begin{aligned} (r'_h)^* \circ \Phi_{\mathfrak{g}}(f)(Z)(g) &= \Phi_{\mathfrak{g}}(f)(\text{Ad}(h^{-1})(Z))(gh) = f(\gamma_{\mathfrak{g}}(\text{Ad}(h^{-1})(Z)))(gh) = \\ &= f(\text{Ad}(h^{-1})(\gamma_{\mathfrak{g}}(Z)))(gh) = r_h^*(f)(\gamma_{\mathfrak{g}}(Z))(g) = \Phi_{\mathfrak{g}} \circ r_h^*(f)(Z)(g). \end{aligned}$$

Similarly, we have $(l'_h)^* \circ \Phi_{\mathfrak{g}} = \Phi_{\mathfrak{g}} \circ l_h^*$. \square

Lemma 4. *Let (Z_j) and (X_i) be bases of even and odd left invariant vector fields respectively, ε^i and X_i be as in Example 3. Then any split grading operator w on \mathcal{G} , which is invariant with respect to the left translations, has the following form*

$$w = \sum \varepsilon^i X_i + \sum f^i X_i + \sum g^j Z_j,$$

where f^i are odd polynomials in ε^i with constant coefficient of degree greater than 3, and g^j are even polynomials in ε^i with constant coefficient of degree greater than 2.

Proof. By Proposition 2, any grading operator has the following form

$$w = \sum \varepsilon^i X_i + \chi, \quad \text{where } \chi = \sum f^i X_i + \sum g^j Z_j,$$

$f_i \in H^0(\mathcal{G}_0, (\mathcal{O}_{\mathcal{G}})_{(3)})$ and $g^j \in H^0(\mathcal{G}_0, (\mathcal{O}_{\mathcal{G}})_{(2)})$. We have:

$$l_g^* \circ w = \sum l_g^*(\varepsilon^i) X_i \circ l_g^* + \sum l_g^*(f^i) X_i \circ l_g^* + \sum l_g^*(g^j) Z_j \circ l_g^* = w \circ l_g^*.$$

By definition, $l_g^*(\varepsilon^i) = \varepsilon^i$. Therefore, $l_g^*(f^i) = f^i$ and $l_g^*(g^j) = g^j$. Let us decompose f^i and g^j using the coordinates ε^i and the sheaf homomorphism $\Phi_{\mathfrak{g}}$:

$$\begin{aligned} f^i &= f_{klr}^i \varepsilon^k \varepsilon^l \varepsilon^r + \dots, \\ g^j &= g_{kl}^j \varepsilon^k \varepsilon^l + \dots, \end{aligned}$$

where f_{klr}^i, \dots and $g_{kl}^j, \dots \in \Phi_{\mathfrak{g}}^{-1}(\mathcal{F}_{\mathcal{G}_0})$ and $\mathcal{F}_{\mathcal{G}_0} = \text{Hom}(1, \mathcal{F}_{\mathcal{G}_0})$. The sheaf $\mathcal{F}_{\mathcal{G}_0} = \text{Hom}(1, \mathcal{F}_{\mathcal{G}_0})$ is obviously invariant with respect to the left translations, see (4). By Lemma 3, the sheaf $\Phi_{\mathfrak{g}}^{-1}(\mathcal{F}_{\mathcal{G}_0})$ is also invariant with respect to the left translations. Therefore, $\Phi_{\mathfrak{g}}(f_{klr}^i), \dots$ and $\Phi_{\mathfrak{g}}(g_{kl}^j), \dots$ are left invariant functions on \mathcal{G}_0 , hence, constants. The proof is complete. \square

It follows from the proof of Lemma 4 that we may identify the space of vector fields, which are invariant with respect to all left translations $l_g, g \in \mathcal{G}_0$, with $H^0(\text{pt}, \mathcal{O}_{\mathcal{G} \setminus \mathcal{G}_0}) \otimes \mathfrak{g}$. Furthermore, the Lie supergroup \mathcal{G} (and hence any Lie subsupergroup \mathcal{H} of \mathcal{G}) acts on the supermanifold $\mathcal{G} \setminus \mathcal{G}_0$ on the right. In other words, $H^0(\text{pt}, \mathcal{O}_{\mathcal{G} \setminus \mathcal{G}_0})$ is a \mathcal{G} -module. The Lie superalgebra \mathfrak{g} is also a \mathcal{G} -module: the action is given by the following action of the corresponding Harish-Chandra pair (G, \mathfrak{g}) :

$$g \mapsto (X \mapsto r_g^* \circ X \circ (r_g^{-1})^*); \quad Y \mapsto (X \mapsto [Y, X]),$$

where $g \in \mathcal{G}_0, X, Y \in \mathfrak{g}$ and we identify \mathfrak{g} with the space of all left invariant vector fields. If \mathcal{H} is a Lie subsupergroup of \mathcal{G} and $\mathfrak{h} = \text{Lie } \mathcal{H}$ then $\mathfrak{g}/\mathfrak{h}$ is a \mathcal{H} -module.

Lemma 5. *Let w be as in Lemma 4. The vector field w satisfies (11) iff*

$$\overline{w} \in (H^0(\text{pt}, \mathcal{O}_{\mathcal{G} \setminus \mathcal{G}_0}) \otimes \mathfrak{g}/\mathfrak{h})^{\mathcal{H}}, \quad (13)$$

where \overline{w} is the image of w by the natural mapping

$$H^0(\text{pt}, \mathcal{O}_{\mathcal{G} \setminus \mathcal{G}_0}) \otimes \mathfrak{g} \rightarrow H^0(\text{pt}, \mathcal{O}_{\mathcal{G} \setminus \mathcal{G}_0}) \otimes \mathfrak{g}/\mathfrak{h}.$$

Proof. Let $\overline{w} \in (H^0(\text{pt}, \mathcal{O}_{\mathcal{G} \setminus \mathcal{G}_0}) \otimes \mathfrak{g}/\mathfrak{h})^{\mathcal{H}}$. It follows that

$$r_h^* \circ w \circ (r_h^{-1})^* - w \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G} \setminus \mathcal{G}_0}) \otimes \mathfrak{h}, \quad h \in H,$$

and

$$[Y, w] \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G} \setminus \mathcal{G}_0}) \otimes \mathfrak{h}, \quad Y \in \mathfrak{h}.$$

Hence, the conditions (11) are satisfied.

On the other hand, if the conditions (11) are satisfied, then the vector fields $r_h^* \circ w \circ (r_h^{-1})^* - w$ and $[Y, w]$ are vertical with respect to the projection $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}$. Therefore, $r_h^* \circ w \circ (r_h^{-1})^* - w$ and $[Y, w]$ belong to $H^0(\text{pt}, \mathcal{O}_{\mathcal{G}/\mathcal{G}_0}) \otimes \mathfrak{h}$. \square

Using the language of super Harish-Chandra pairs we may rewrite the condition (11) in the following way:

$$\overline{w} \in (H^0(\text{pt}, \mathcal{O}_{\mathcal{G}/\mathcal{G}_0}) \otimes \mathfrak{g}/\mathfrak{h})^{\mathcal{H}_0}, \quad \overline{w} \in (H^0(\text{pt}, \mathcal{O}_{\mathcal{G}/\mathcal{G}_0}) \otimes \mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}.$$

Our aim is now to describe the space $(H^0(\text{pt}, \mathcal{O}_{\mathcal{G}/\mathcal{G}_0}) \otimes \mathfrak{g}/\mathfrak{h})^{\mathcal{H}_0}$. More precisely, we will show that $H^0(\text{pt}, \mathcal{O}_{\mathcal{G}/\mathcal{G}_0}) \simeq (\bigwedge \mathfrak{g}_{\bar{1}})^*$ as \mathcal{G}_0 -modules. Indeed, it follows from (4) that the vector space $(\bigwedge \mathfrak{g}_{\bar{1}})^* \subset \text{Hom}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0}) = \mathcal{O}_{\text{gr } \mathcal{G}}$ consists of all functions which are invariant with respect to the left translations. Therefore, $\Phi_{\mathfrak{g}}$ induces the isomorphism between $H^0(\text{pt}, \mathcal{O}_{\mathcal{G}/\mathcal{G}_0})$ and $(\bigwedge \mathfrak{g}_{\bar{1}})^*$. Assume now that $\varepsilon^i = \Phi_{\mathfrak{g}}^{-1}(e^i)$, where (e^i) is a certain basis on $\mathfrak{g}_{\bar{1}}^*$. Denote by L the linear combinations of ε^i . It follows from Lemma 3 that L is \mathcal{G}_0 -invariant and it is isomorphic to $\mathfrak{g}_{\bar{1}}^*$ via isomorphism $\Phi_{\mathfrak{g}}$ as \mathcal{G}_0 -modules. Hence, we have

$$\begin{aligned} H^0(\text{pt}, \mathcal{O}_{\mathcal{G}/\mathcal{G}_0}) \otimes \mathfrak{g} &\simeq \bigwedge(\mathfrak{g}_{\bar{1}}^*) \otimes \mathfrak{g} \text{ as } \mathcal{G}_0\text{-modules,} \\ H^0(\text{pt}, \mathcal{O}_{\mathcal{G}/\mathcal{G}_0}) \otimes \mathfrak{g}/\mathfrak{h} &\simeq \bigwedge(\mathfrak{g}_{\bar{1}}^*) \otimes \mathfrak{g}/\mathfrak{h} \text{ as } \mathcal{H}_0\text{-modules.} \end{aligned}$$

The \mathcal{G}_0 -module $\bigwedge(\mathfrak{g}_{\bar{1}}^*) \otimes \mathfrak{g}$ has a natural \mathcal{G}_0 -invariant: the identity operator from $\mathfrak{g}_{\bar{1}}^* \otimes \mathfrak{g}$. The corresponding to this invariant vector field is $\sum \varepsilon^i X_i$. The result of our study is:

Theorem 1. *The following conditions are equivalent:*

1. w is the split grading operator corresponding to a left invariant grading on \mathcal{G} , which is compatible with $\mathcal{O}_{\mathcal{G}/\mathcal{H}}$.
2. $w = \sum \varepsilon^i X_i + \chi$, where $\sum \varepsilon^i X_i$ is described above, $\chi \in H^0(\mathcal{G}_0/\mathcal{H}_0, \mathcal{T}_{(2)\bar{0}})$, such that

$$\overline{\chi} \in \left(\bigoplus_{p \geq 2} \bigwedge^p (L) \otimes \mathfrak{g}/\mathfrak{h} \right)_{\bar{0}}^{\mathcal{H}_0} \simeq \left(\bigoplus_{p \geq 2} \bigwedge^p (\mathfrak{g}_{\bar{1}})^* \otimes \mathfrak{g}/\mathfrak{h} \right)_{\bar{0}}^{\mathcal{H}_0}, \quad (14)$$

$$\overline{w} \in \left(\bigwedge^p (L) \otimes \mathfrak{g}/\mathfrak{h} \right)^{\mathfrak{h}}. \quad (15)$$

Recall that \overline{w} is the image of w by the natural mapping

$$H^0(\text{pt}, \mathcal{O}_{\mathcal{G}/\mathcal{G}_0}) \otimes \mathfrak{g} \rightarrow H^0(\text{pt}, \mathcal{O}_{\mathcal{G}/\mathcal{G}_0}) \otimes \mathfrak{g}/\mathfrak{h}.$$

5. An application

As above \mathcal{G} is a Lie supergroup and $\mathcal{H} \subset \mathcal{G}$ is a Lie subsupergroup, \mathfrak{g} and \mathfrak{h} are the Lie superalgebras of \mathcal{G} and \mathcal{H} respectively, and $\mathcal{M} := \mathcal{G}/\mathcal{H}$. Consider the map $\rho : \mathfrak{g}_{\bar{0}} \rightarrow H^0(\text{pt}, \mathcal{T}_{\mathcal{G}/\mathcal{H}})$. Let us describe its kernel. We have for $X \in \mathfrak{g}_{\bar{0}}$ and $f \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}/\mathcal{H}})$:

$$X(f)(Y)(e) = \frac{d}{dt}\bigg|_{t=0} f(\text{Ad}(\exp(tX))Y)(\exp(tX)) = \frac{d}{dt}\bigg|_{t=0} f(\text{Ad}(\exp(tX))Y)(e),$$

where $Y = Y_1 \cdots Y_r$, $Y_i \in \mathfrak{g}_{\bar{1}}$ and t is an even parameter. Hence,

$$\text{Ker } \rho = \text{Ker}(\text{ad}|_{\mathfrak{g}_{\bar{1}}}),$$

where ad is the adjoint representation of $\mathfrak{g}_{\bar{0}}$ in \mathfrak{g} .

Furthermore, denote

$$\begin{aligned} A &:= \text{Ker}(\mathcal{G}_0 \ni g \mapsto \bar{l}_g : \mathcal{G}/\mathcal{H} \rightarrow \mathcal{G}/\mathcal{H}); \\ \mathfrak{a} &:= \text{Ker}(\mathfrak{g} \ni X \mapsto H^0(\mathcal{G}_0/\mathcal{H}_0, \mathcal{T}_{\mathcal{G}/\mathcal{H}})), \end{aligned}$$

where \bar{l}_g is the automorphism of \mathcal{G}/\mathcal{H} induced by the left translation l_g . The pair (A, \mathfrak{a}) is a super Harish-Chandra pair. An action of \mathcal{G} on \mathcal{M} is called *effective* if the corresponding to (A, \mathfrak{a}) Lie supergroup is trivial.

Theorem 2. *Assume that the action of \mathcal{G} on \mathcal{M} is effective. If $[\mathfrak{g}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] \subset \mathfrak{h}_{\bar{0}} \cap \text{Ker}(\text{ad}|_{\mathfrak{g}_{\bar{1}}})$, then \mathcal{M} is split.*

Proof. Let us show that in this case the vector field $w = v + 0 = \sum \varepsilon^i X_i$ from Theorem 1 is a (left invariant) split grading operator on \mathcal{M} using Theorem 1. The condition (14) satisfies trivially, because $\chi = 0$. Let us check the condition (15). We have:

$$[Y, v] = \sum Y(\varepsilon^i)X_i - \sum \varepsilon^i[Y, X_i].$$

Since $[\mathfrak{g}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] \subset \mathfrak{h}_{\bar{0}}$, we get $\sum \varepsilon^i[Y, X_i] \in \bigwedge^p(L) \otimes \mathfrak{h}$. Hence, we have to show that $\sum Y(\varepsilon^i)X_i \in \bigwedge^p(L) \otimes \mathfrak{h}$.

We need the following well-known formula:

$$Y(f)(X)(g) = (-1)^{p(Y)} f(XY)(g),$$

where $X \in \mathfrak{U}(\mathfrak{g})$, $Y \in \mathfrak{g}$, $g \in \mathcal{G}_0$, $f \in \mathcal{O}_{\mathcal{G}}$ and $p(Y)$ is the parity of Y . Denote by \mathcal{P} the Lie supergroup with the following super Harish-Chandra pair $(\mathcal{P}_0, \mathfrak{p})$:

$$\begin{aligned} \mathcal{P}_0 &:= \text{Ker}(\mathcal{G}_0 \ni g \mapsto \bar{r}_g : \mathcal{G} \backslash \mathcal{G}_0 \rightarrow \mathcal{G} \backslash \mathcal{G}_0); \\ \mathfrak{p} &:= \text{Ker}(\mathfrak{g} \ni X \mapsto H^0(\text{pt}, \mathcal{T}_{\mathcal{G} \backslash \mathcal{G}_0})), \end{aligned}$$

where \bar{r}_g is the automorphism of $\mathcal{G} \setminus \mathcal{G}_0$ induced by the left translation r_g . The Lie supergroup $\mathcal{G} \setminus \mathcal{P}$ acts on $(\text{pt}, \mathcal{O}_{\mathcal{G} \setminus \mathcal{G}_0})$ transitively. Hence, we may assume that $\mathcal{O}_{\mathcal{G} \setminus \mathcal{G}_0}$ is a subsheaf of $\mathcal{O}_{\mathcal{G} \setminus \mathcal{P}}$. Denote by \tilde{X} the image of X by the map $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{p}$. By Lemma 1, we have for $Y \in \mathfrak{h}_{\bar{1}}$ and $X_i \in \mathfrak{g}_{\bar{1}}$:

$$\begin{aligned} \tilde{Y}(\varepsilon^i)(\gamma_{\mathfrak{g}}(\tilde{X}_1 \wedge \cdots \wedge \tilde{X}_r))(g) &= -\varepsilon^i(\gamma_{\mathfrak{g}}(\tilde{X}_1 \cdots \tilde{X}_r)\tilde{Y})(g) = \\ -\varepsilon^i(\gamma_{\mathfrak{g}}(\tilde{X}_1 \wedge \cdots \wedge \tilde{X}_r \wedge \tilde{Y}))(g) &= -e^i(\tilde{X}_1 \wedge \cdots \wedge \tilde{X}_r \wedge \tilde{Y})(g). \end{aligned}$$

Here we use the fact that \tilde{X}_i and \tilde{Y} commute by assumption that the commutator $[\mathfrak{g}_{\bar{1}}, \mathfrak{h}_{\bar{1}}]$ is contained in $\text{Ker}(\mathfrak{g}_{\bar{0}} \rightarrow H^0(\text{pt}, \mathcal{T}_{\mathcal{G} \setminus \mathcal{G}_0}))$. Hence,

$$Y(\varepsilon^i) = -e^i(Y) = Y'(e^i),$$

where Y' is the corresponding to Y left invariant vector field in $\text{gr } \mathcal{G}$. Therefore, we have

$$\sum Y(\varepsilon^i)X_i = -\sum e^i(Y)X_i = Y \in \bigwedge^p(L) \otimes \mathfrak{h}.$$

The proof is complete. \square

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